LIE GROUPS AND LIE ALGEBRAS MIDTERM EXAMINATION

Total marks: 60.

(1) What is a matrix Lie group? Define the groups $SO(n)$, $SU(n)$ and $Sp(n)$. Prove that $SO(n)$ is connected. (2+4+6 marks)

(2) Define the exponential of a $n \times n$ complex matrix. Prove that the series in this definition is absolutely convergent. Calculate the exponential of the 2×2 matrix

$$\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

(2+4+6 marks)

(3) Define the Lie algebra of a matrix Lie group. Calculate the Lie algebras of the matrix Lie groups $SO(n)$, $SU(n)$ and $Sp(n)$. (2+4+4+4 marks)

(4) Show that the exponential map for the Lie group $U(n)$ is surjective but not injective. Give an example of a matrix Lie group G such that the exponential map for G is not surjective. (6+4 marks)

(5) Let G be matrix Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Let H be the (unique) connected Lie subgroup of G whose Lie algebra is $\mathfrak{h}$. Let K be a simply-connected compact matrix Lie group and supoose the Lie algbra of K is isomorphic to $\mathfrak{h}$. Prove that H is a closed subgroup of G. Is H necessarily isomorphic to K? (8+4 marks)

Date: September 8, 2014.